J Math Chem (2010) 48:66–71
DOI 10.1007/s10910-009-9624-3

ORIGINAL PAPER

Positive entropy of a coupled lattice system related with Belusov-Zhabotinskii reaction

Juan Luis García Guirao · Marek Lampart

Received: 4 August 2009 / Accepted: 29 October 2009 / Published online: 12 November 2009

Abstract In this paper we present a lattice dynamical system stated by Kaneko in (Phys Rev Lett, 65: 1391–1394, 1990) which is related to the Belusov-Zhabotinskii reaction. We prove that this CML (Coupled Map Lattice) system has positive topological entropy for zero coupling constant.

Keywords Coupled map lattice · Positive topological entropy

1 Introduction

Classical Discrete Dynamical Systems (DDS's), i.e., a couple composed by a space X (usually compact and metric) and a continuous self-map ψ on X, have been highly considered in the literature (see e.g., [3] or [9]) because are good examples of problems coming from the theory of Topological Dynamics and model many phenomena from biology, physics, chemistry, engineering and social sciences (see for example,

This research was supported in part by MEC (Ministerio de Educación y Ciencia, Spain) grants MTM2005-03860 and MTM2005-06098-C02-01; Fundación Séneca (Comunidad Autónoma de la Región de Murcia), grant 00684-FI-04, by Grant Agency of the Czech Republic grant 201/07/P032 and the Ministry of Education of the Czech Republic No. MSM6198910027.

J. L. G. Guirao (✉)
Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Hospital de Marina, 30203 Cartagena, Región de Murcia, Spain
e-mail: juan.garcia@upct.es

M. Lampart
Department of Applied Mathematics, VŠB - Technical University of Ostrava, 17. listopadu 15/2172, 708 33 Ostrava, Czech Republic
e-mail: marek.lampart@vsb.cz

[8,16,19] or [18]). In most cases in the formulation of such models ψ is a C^∞, an analytical or a polynomial map.

Coming from chemical engineering applications, such a digital filtering, imaging and spatial vibrations of the elements which compose a given chemical product, a generalization of DDS's have recently appeared as an important subject for investigation, we mean the so called *Lattice Dynamical Systems* or *1d Spatiotemporal Discrete Systems*. In the next section we provide all the definitions. To show the importance of these type of systems, see for instance [6].

To analyze when one of these type of systems have a complicated dynamics or not by the observation of one topological dynamics property is an open problem. The aim of the present paper is to show by using the notion of *topological entropy* to characterize the dynamical complexity of coupled lattice systems stated by K. Kaneko in [15] (for more details see for references therein) which is related to the Belusov-Zhabotinskii reaction showing the existence of positive topological entropy for zero coupling constant. We present some other problems for the future related with chemical applications.

Let us recall the notion of Positive topological entropy which is known to topological chaos.

An attempt to measure the complexity of a dynamical system is based on a computation of how many points are necessary in order to approximate (in some sense) with their orbits all possible orbits of the system. A formalization of this intuition leads to the notion of topological entropy of the map f, which is due to Adler, Konheim and McAndrew [1]. We recall here the equivalent definition formulated by Bowen [5], and independently by Dinaburg [10]: the *topological entropy* of a map f is a number $h(f) \in [0, \infty]$ defined by

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \# E(n, f, \varepsilon),$$

where $E(n, f, \varepsilon)$ is a (n, f, ε)–span with minimal possible number of points, i.e., a set such that for any $x \in \mathbb{X}$ there is $y \in E(n, f, \varepsilon)$ satisfying $d\left(f^j(x), f^j(y)\right) < \varepsilon$ for $1 \leq j \leq n$.

A map f is *topologically chaotic* (briefly, PTE) if its topological entropy $h(f)$ is positive.

2 Notation and basic construction

The state space of LDS (Lattice Dynamical System) is the set

$$\mathcal{X} = \{x |\ x = \{x_i\},\ \ x_i \in \mathbb{R}^d,\ \ i \in \mathbb{Z}^D,\ \ \| x_i \| < \infty\},$$

where $d \geq 1$ is the dimension of the range space of the map of state x_i, $D \geq 1$ is the dimension of the lattice and the l^2 norm $\| x \|_2 = (\Sigma_{i \in \mathbb{Z}^D} \mid x_i \mid^2)^{1/2}$ is usually taken ($\mid x_i \mid$ is the length of the vector x_i).

We deal with the following 1d-LD CML (Coupled Map Lattice) system which was stated by Kaneko in [15] (for more details see for references therein) and it is related to the Belusov-Zhabotinskii reaction (see [16] and for experimental study of chemical turbulence by this method [12–14]):

$$x_n^{m+1} = (1-\epsilon) f\left(x_n^m\right) + \epsilon/2\left[f\left(x_{n-1}^m\right) - f\left(x_{n+1}^m\right)\right], \tag{1}$$

where m is discrete time index, n is lattice side index with system size L (i.e. $n = 1, 2, \ldots L$), ϵ is coupling constant and $f(x)$ is the *unimodal map* on the unite closed interval $I = [0, 1]$, i.e. $f(0) = f(1) = 0$ and f has unique critical point c with $0 < c < 1$ such that $f(c) = 1$. For simplicity we will deal with so called "tent map", defined by

$$f(x) = \begin{cases} 2x, & x \in [0, 1/2), \\ 2 - 2x, & x \in [1/2, 1]. \end{cases}$$

In general, one for the following periodic boundary conditions of the system (1) is assumed:

(1) $x_n^m = x_{n+L}^m$,
(2) $x_n^m = x_n^{m+L}$,
(3) $x_n^m = x_{n+L}^{m+L}$,

standardly, the first case of the boundary conditions is used.

The Eq. (1) was studied by many authors, mostly experimentally or semi-analytically then analytically. The first paper with analytic results is [7], where it was proved that this system is chaotic in the sense of Li and Yorke.

We consider, as an example the 2-element one-way coupled logistic lattice (OCLL, see [17]) $H : I^2 \to I^2$ written as

$$\begin{aligned} x_1^{m+1} &= (1-\epsilon) f(x_1^m) + \epsilon f(x_2^m), \\ x_2^{m+1} &= \epsilon f(x_1^m) + (1-\epsilon) f(x_2^m), \end{aligned} \tag{2}$$

where f is the tent map.

The following construction is similar to the [4]. Since the critical point for the tent map is equal to 1/2 we can divide the interval I into two sets $P_1 = [0, 1/3)$ and $P_2 = (2/3, 1]$ and get a family $\mathcal{P} = \{P_1, P_2\}$. Then each point $x_0 \in \Lambda_1$ can be represented as an infinite symbol sequence $C_1(x_0) = \alpha = a_1 a_2 a_3 \ldots$ where Λ_1 is Cantor ternary set and

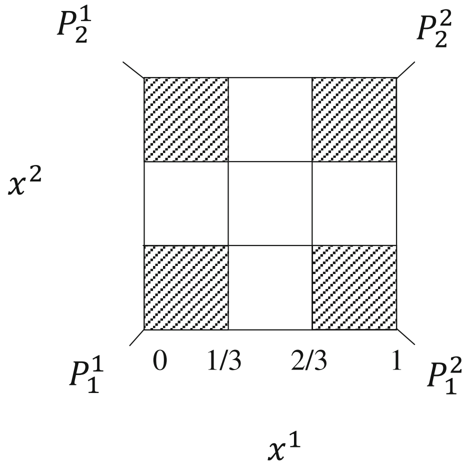
$$a_n = \begin{cases} 0 & \text{if} \quad f^n(x_0) \in P_1, \\ 1 & \text{if} \quad f^n(x_0) \in P_2. \end{cases}$$

Returning to (2) we can divide its range set into four sets $\mathcal{P} = \{P_1^1, P_2^1, P_1^2, P_2^2\}$ (see the figure below) where the upper index corresponds to the x_1 coordinate and x_2 to the lower one. Then again each point $p \in \Lambda_2$ can be encrypted as an infinite symbol sequence $C_2(p) = \alpha = a_1 a_2 a_3 \ldots$ where Λ_2 is 2-dimensional Cantor ternary set[1]

[1] By *n-dimensional Cantor set* we mean the Cantor set constructed as subset of $\mathbb{R}^n$.

and

$$a_n = \begin{cases} 0 & \text{if} \quad H^n(p) \in P_1^1, \\ 1 & \text{if} \quad H^n(p) \in P_2^1, \\ 2 & \text{if} \quad H^n(p) \in P_1^2, \\ 3 & \text{if} \quad H^n(p) \in P_2^2. \end{cases}$$



Now, we denote the *k-shift* operator σ_k on k symbol alphabet, defined by $\sigma_k : \Sigma_k \to \Sigma_k$ and $\sigma_k(a_1a_2a_3\ldots) = a_2a_3\ldots$ where $\Sigma_k = \{\alpha \mid \alpha = a_1a_2a_3\ldots \text{ and } a_i \in \{1, 2, \ldots k\}\}$, so the effect of this operator is to delete the first symbol of the sequence α.

We can observe that Λ_2 is invariant[2] subset of the range space of the system (2) and that each its point is encoded by exactly one point from Σ_4, for $\epsilon = 0$. So, by [11] the shift operator σ_4 acts on Σ_4 exactly as (2) on Λ_2, for $\epsilon = 0$.

3 Main result

We say that two dynamical systems (X, f) and (Y, g) are *topologically conjugated* if there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$ (the diagram commutes), such homeomorphism is called *conjugacy*. Then:

Proposition 1 [20] *If (X, f) and (Y, g) are topologically conjugated systems then $h(f) = h(g)$.*

For the proof of the main result we also use well known result:

Proposition 2 [20] *Let σ_k be the k-shift operator. Then $h(\sigma_k) = k \log 2$.*

[2] A set M is invariant for the map f if $f(M) \subset M$.

Theorem 1 *The system*

$$x_n^{m+1} = (1-\epsilon) f\left(x_n^m\right) + \epsilon/2\left[f\left(x_{n-1}^m\right) - f\left(x_{n+1}^m\right)\right],$$

has positive topological entropy for $\epsilon = 0$. *Moreover, its entropy equals to* $L \log 2$.

Proof By the construction of the Sect. 2 it follows that the 2–dimensional system (1) contains 2-dimensional Cantor set which is conjugated (see, e.g. [11]) to the shift space Σ_4 by the conjugacy map C_2, for $\epsilon = 0$. Then by Proposition 1 the system has topological entropy equal to the entropy of σ_4. Consequently, by Proposition 2 its entropy is $2 \log 2$.

To the end of the proof, it suffice to note, that the construction of the Sect. 2 can be generalized to the L-dimensional systems. Such system will be conjugated to the 2^L-shift by C_L conjugacy and by the same arguments, as in the paragraph above, its entropy equals to $L \log 2$. □

4 Concluding remarks

The proof of the main result can be done in an alternative way. For zero coupling constant it is obvious that each lattice side contains a subsystem conjugated to (Σ_2, σ_2). Then the system (1) contains subsystem conjugated to the L-times product of (Σ_2, σ_2) and by $h(\underbrace{\sigma_2 \times \cdots \times \sigma_2}_{L}) = Lh(\sigma_2)$ (see, e.g. [20]) the assertion follows.

For non-zero coupling constants the dynamical behaviour of the system (1) is more complicated. The first question is how the invariant subsets of phase space looks like? Secondly, is the set of periodic points dense in the range space? The answer for this question will be nontrivial. Similar system was studied in [2] and there was used the method of resultants to prove existence of periodic points of higher order. The same concept like in [2] should be used.

References

1. R.L. Adler, A.G. Konheim, M.H. McAndrew, Topological entropy. Trans. Am. Math. Soc. **114**, 309–319 (1965)
2. F. Balibrea, J.L. García Guirao, M. Lampart, J. Llibre, Dynamics of a Lotka-Volterra map. Fund. Math. **191**(3), 265–279 (2006)
3. L.S. Block, W.A. Coppel, *Dynamics in One Dimension, Springer Monographs in Mathematics* (Springer, New York, 1992)
4. E. Bollt, N.J. Corron, S.D. Pethel, Symbolic dynamics of coupled map lattice. Phys. Rew. Lett. **96**, 1–4 (2006)
5. R. Bowen, Entropy for group endomorphisms and homogeneous spaces. Trans. Am. Math. Soc. **153**, 401–414 (1971)
6. J.R. Chazottes, B. Fernndez, Dynamics of Coupled Map Lattices and of Related Spatially Extended Systems. Lecture Notes in Physics, **671**, 2005
7. G. Chen, S.T. Liu, On spatial periodic orbits and spatial chaos. Int. J. Bifur. Chaos **13**, 935–941 (2003)
8. R.A. Dana, L. Montrucchio, Dynamical complexity in duopoly games. J. Econom. Theory **40**, 40–56 (1986)
9. R.L. Devaney, *An Introduction to Chaotics Dynamical Systems* (Benjamin/Cummings, Menlo Park, CA, 1986)

10. E.I. Dinaburg, A connection between various entropy characterizations of dynamical systems. Izv. Akad. Nauk SSSR Ser. Mat. **35**, 324–366 (1971)
11. H. Furnsterbeg, *Recurrence in Ergodic Theory and Combinational Number Theory* (Princeton University Press XI, Princeton, New Jersey, 1981)
12. K. Hirakawa, Y. Oono, H. Yamakazi, Experimental study on chemical turbulence. II. J. Phys. Soc. Jap. **46**, 721–728 (1979)
13. J.L. Hudson, K.R. Graziani, R.A. Schmitz, Experimental evidence of chaotic states in the Belusov-Zhabotinskii reaction. J. Chem. Phys. **67**, 3040–3044 (1977)
14. J.L. Hudson, M. Hart, D. Marinko, An experimental study of multiplex peak periodic and nonperiodic oscilations in the Belusov-Zhabotinskii reaction. J. Chem. Phys. **71**, 1601–1606 (1979)
15. K. Kaneko, Globally coupled chaos violates law of large numbers. Phys. Rev. Lett. **65**, 1391–1394 (1990)
16. M. Kohmoto, T. Oono, Discrete model of chemical turbulence. Phys. Rev. Lett. **55**, 2927–2931 (1985)
17. K. Kaneko, H.F. Willeboordse, Bifurcations and spatial chaos in an open flow model. Phys. Rew. Lett. **73**, 533–536 (1994)
18. B. Vander Pool, Forced oscilations in a circuit with nonlinear resistence. Lond. Edinb. Dublin Phil. Mag. **3**, 109–123 (1927)
19. T. Puu, Chaos in duopoly pricing. Chaos Solitions Fractals **1**, 573–581 (1991)
20. P. Walters, *An Introduction to Ergodic Theory* (Springer, New York, 1982)